

Conformal Quantum Effects and the Anisotropic Singularities of Scalar-Tensor Theories of Gravity

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We show that the inclusion of a term $C_{abcd}C^{abcd}$ in the action can remove the recently described anisotropic singularity occurring on the hypersurface $F(\phi) = 0$ of scalar-tensor theories of gravity of the type

$$S = \int d^4x \sqrt{-g} \{F(\phi)R - \partial_a \phi \partial^a \phi - 2V(\phi)\},$$

preserving, by construction, all of their isotropic solutions. We show that, in principle, a higher order term of this type can arise from considerations about the renormalizability of the semiclassical approach to the theory. Such result brings again into consideration the quintessential models recently proposed based in a conformally coupled scalar field ($F(\phi^2) = 1 - \frac{1}{6}\phi^2$) with potential $V(\phi) = \frac{m}{2}\phi^2 - \frac{\Omega}{4}\phi^4$, that have been discharged as unrealistic precisely by their anisotropic instabilities on the hypersurface $F(\phi) = 0$.

KEY WORDS: cosmology; singularities; quintessence.

1. INTRODUCTION

In Gunzig *et al.* (2001a,b), was proposed a quintessential model corresponding to the homogeneous and isotropic solutions of the cosmological model described by the action:

$$S = \int d^4x \sqrt{-g} \{F(\phi)R - \partial_a \phi \partial^a \phi - 2V(\phi)\}, \quad (1)$$

with $F(\phi) = 1 - \frac{1}{6}\phi^2$, the so-called conformal coupling, and $V(\phi) = \frac{m}{2}\phi^2 - \frac{\Omega}{4}\phi^4$. Some novel and interesting dynamical behaviors were identified: superinflation regimes, a possible avoidance of big-bang singularities through classical birth of the universe from empty Minkowski space, spontaneous entry into and exit

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from inflation, and a cosmological history suitable for describing quintessence. The next natural step was (Abramo *et al.*, 2003a,b) the analysis of the robustness of these results against small perturbations in initial conditions and in the model itself. By considering general coupling $F(\phi)$ models as in Esposito-Farese and Polarski (2001), we generalize previous results (Bertolami, 1987; Deser, 1984; Futamase *et al.*, 1981; Futamase and Maeda, 1989; Hosotani, 1949; Starobinski, 1981) and identify two kinds of dynamically unavoidable singularities in the models described by (1).

The first one appears only in the anisotropic case and corresponds to the hypersurfaces $F(\phi) = 0$. It is a direct generalization of the Starobinski singularities of conformally coupled anisotropic solution (Starobinski, 1981). It implies that the homogeneous and isotropic solutions passing from the $F(\phi) > 0$ to the $F(\phi) < 0$ region in the model described in Gunzig *et al.* (2001a,b) are extremely unstable against anisotropic perturbations, challenging its proposal as a quintessential model. The second type of singularity corresponds to $F_1(\phi) = 0$, with

$$F_1(\phi) = F(\phi) + \frac{3}{2}(F'(\phi))^2, \quad (2)$$

and it is present even for the homogeneous and isotropic cases. Futamase and coworkers (Futamase *et al.*, 1981; Futamase and Maeda, 1989) identified both singularities in the context of chaotic inflation in $F(\phi) = 1 - \xi\phi^2$ theories (See also Bertolami, 1987; Deser, 1984; Hosotani, 1985). The first singularity is always present for $\xi > 0$ and the second one for $0 < \xi < 1/6$. The conclusions of Abramo *et al.* (2003a,b) are, however, more general since we treat the case of general $F(\phi)$ and $V(\phi)$ and our results are based on the analysis of true geometrical invariants. The main result is that the system governed by (1) is *generically* singular on both hypersurfaces $F(\phi) = 0$ and $F_1(\phi) = 0$.

The anisotropic singularity occurring on $F(\phi) = 0$ was the major obstacle in developing the model proposed in Gunzig *et al.* (2001a,b). Note that singularities of the second type are absent in the conformally coupled case. Although the isotropic solutions are always regular on the hypersurface $F(\phi) = 0$, any small deviation of isotropy will have catastrophic consequences, leading to a space-time singularity in a finite time. Even a very small amount of anisotropy will be hugely amplified, feeding the energy content of the scalar field ϕ and increasing the space-time curvature toward a true singularity. Our purpose here is to show that the inclusion of the higher order term $C_{abcd}C^{abcd}$ can eliminate this anisotropic singularity preserving, by construction, all isotropic solutions. Moreover, we will see that a quantum counterterm precisely of this form can arise from considerations about the renormalizability of the semiclassical theory described by (1).

The next section presents a brief review of the geometric nature of the anisotropic singularity. Section 3 discusses the possible appearance of the

conformal counterterm $C^{abcd}C^{abcd}$. Its dynamical implications are presented in Section 4. The last section presents some concluding remarks.

2. THE SINGULARITY

The equations derived from the action (1) are the Klein-Gordon equation

$$\square\phi - V'(\phi) + \frac{1}{2}F'(\phi)R = 0, \tag{3}$$

and the Einstein equations

$$F(\phi)G_{ab} = (1 + F''(\phi))\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}[(1 + 2F''(\phi))\partial_c\phi\partial^c\phi + 2V(\phi)] - F'(\phi)(g_{ab}\square\phi - \nabla_a\phi\nabla_b\phi). \tag{4}$$

We considered the simplest anisotropic homogeneous cosmological model, the Bianchi type I, whose spatially flat metric is given by

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2. \tag{5}$$

The dynamically relevant quantities here are $H_i = \dot{a}_i/a$, $i = 1, 2, 3$. For such a metric and a homogeneous scalar field $\phi = \phi(t)$ Eq. (4) can be written as

$$F(\phi)G_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) - F'(\phi)(H_1 + H_2 + H_3)\dot{\phi}, \tag{6}$$

$$\frac{1}{a_1^2}F(\phi)G_{11} = \frac{1 + 2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_1\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right), \tag{7}$$

$$\frac{1}{a_2^2}F(\phi)G_{22} = \frac{1 + 2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_2\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right), \tag{8}$$

$$\frac{1}{a_3^2}F(\phi)G_{33} = \frac{1 + 2F''(\phi)}{2}\dot{\phi}^2 - V(\phi) - F'(\phi)\left(H_3\dot{\phi} + V'(\phi) - \frac{F'(\phi)}{2}R\right). \tag{9}$$

It is quite simple to show that Eqs. (7)–(9) are not compatible, in general, on the hypersurface $F(\phi) = 0$. Subtracting (8) and (9) from (7) we have, on such hypersurface, respectively,

$$F'(\phi)(H_1 - H_2)\dot{\phi} = 0, \text{ and } F'(\phi)(H_1 - H_3)\dot{\phi} = 0. \tag{10}$$

Hence, they cannot be fulfilled in general for anisotropic metrics. This is the origin of the anisotropic singularity. Using the new dynamical variables

$p = H_1 + H_2 + H_3$, $q = H_1 - H_2$, and $r = H_1 - H_3$, Einstein Equations can be cast in the form

$$E(\phi, \dot{\phi}, p, q, r) = -\frac{1}{3}F(\phi)(p^2 + qr - q^2 - r^2) + \frac{\dot{\phi}^2}{2} + V(\phi) - pF'(\phi)\dot{\phi} = 0, \quad (11)$$

$$\dot{q} = -\left(p + \frac{F'(\phi)}{F(\phi)}\dot{\phi}\right)q, \quad (12)$$

$$\dot{r} = -\left(p + \frac{F'(\phi)}{F(\phi)}\dot{\phi}\right)r, \quad (13)$$

$$\begin{aligned} -2F_1(\phi)\dot{p} &= (F(\phi) + 2F'(\phi)^2)p^2 + \frac{3}{2}(1 + 2F''(\phi))\dot{\phi}^2 - 3V(\phi) \\ &\quad - 3F'(\phi)V'(\phi) - p\dot{\phi}F'(\phi) + (F(\phi) + F'(\phi)^2)(q^2 + p^2 - qr) \end{aligned} \quad (14)$$

A closer analysis of Eqs. (12)–(13) reveals the presence of the singularity. In general, the right-hand side of these equations diverge for $F(\phi) = 0$. One can check that this divergence is indeed related to real geometrical singularity by considering the Kretschman scalar $I = R_{abcd}R^{abcd}$ (Abramo *et al.*, 2003a,b). Furthermore, it is dynamically unavoidable since the hypersurface $F(\phi) = 0$ has always an attractive side.

3. RENORMALIZATION AND QUANTUM COUNTERTERMS

The idea of incorporating vacuum semiclassical effects into gravity has a long history, and a good set of references is presented in Birrel and Davies (1982) and Buchbinder *et al.* (1992). Zeldovich was the first to propose (Zeldovich, 1967), in 1967, that a cosmological constant term could arise from quantum considerations of matter. Yet in the sixties, in a set of seminal works, Parker considered (Parker, 1968, 1969, 1971) the effect of the creation of particles in an expanding universe, and discussed the possible backreaction, opening the discussion of anisotropy damping and avoidance of the initial singularity due to quantum corrections (Fischetti *et al.*, 1979; Hartle and Hu, 1979, 1980; Hu and Parker, 1978a,b).

A semiclassical treatment of the model described by (1) with $F(\phi) = 1 - \xi\phi^2$ and $V(\phi) = \frac{m}{2}\phi^2 - \frac{\Omega}{4}\phi^4$, where, hereafter, by semiclassical one means that ϕ is quantized on a classical gravitational background, requires the inclusion of higher order counterterms to ensure the renormalization of the theory. These terms are (Birrel and Davies, 1982; Buchbinder *et al.*, 1992)

$$S_{\text{vac}} \int d^4x \sqrt{-g} (\alpha_1 R^2 + \alpha_2 R_{ab}R^{ab} + \alpha_3 R_{abcd}R^{abcd} + \alpha_4 \square R). \quad (15)$$

The quantum divergences of the semiclassical theory can be removed by the renormalization of the constants $\alpha_{1,2,3,4}$ and the Newtonian constant G . In fact, the full set of quantities affected by the renormalization include the matter field, its mass m and selfcoupling constant Ω , the nonminimal coupling constant ξ , and yet a cosmological constant. All these quantities are, in principle, subject to some quantum running, and indeed some models to describe the reacceleration of the universe have been recently proposed based on vacuum quantum effects (Parker and Raval, 1999a,b, 2000, 2001, 2003a,b,c; Parker and Vanzella, 2003; Shapiro *et al.*, 2003; Shapiro and Sola, 2002).

The last counterterm in (15) does not contribute to the classical dynamics, since it is merely a total divergence. In four dimensions, we have

$$C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2. \tag{16}$$

Hence, it is possible, in principle, to combine α_1 , α_2 , and α_3 in order to have the desired counterterm. We leave the issue of the naturalness of this finely tuned choice to the last section. The Weyl tensor C_{abcd} vanishes identically for isotropic spacetimes, and hence any contribution from this counterterm would affect only the anisotropic case by construction, preserving all isotropic solutions. The task of calculating the variations of the conformal counterterm $C_{abcd}C^{abcd}$ with respect to the metric is simplified if one recalls that in four dimensions the Gauss-Bonnet term

$$E = R_{abcd}R^{abcd} + R^2 - R_{ab}R^{ab} \tag{17}$$

has identically vanishing Euler-Lagrange equations and, hence, does not contribute to the classical dynamics too, implying that the conformal counterterm is dynamically equivalent to the term $R_{abcd}R^{abcd}/2 - R^2/6$. Thus, with the inclusion of the conformal counterterm in the model proposed in Gunzig *et al.* (2001a,b), the resulting dynamics are governed by the action

$$S = \int d^4x \sqrt{-g} \left\{ \left(1 - \frac{1}{6}\phi^2 \right) R + \alpha \left(\frac{1}{2}R_{abcd}R^{abcd} - \frac{1}{6}R^2 \right) - \partial_a\phi\partial^a\phi - 2V(\phi) \right\}, \tag{18}$$

Where $V(\phi) = \frac{m}{2}\phi^2 - \frac{\Omega}{4}\phi^4$ and α is a parameter typically small when compared to $1/G$.

4. THE DYNAMICS

We study here the dynamics governed by the action (18). The Klein-Gordon equation (3) is not affected by the new term. It is clear, however, that new higher

order terms will appear in the left hand side of Einstein equations (6)–(9). The new terms come from the tensors

$$Q_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^4x \sqrt{-g} R_{abcd} R^{abcd} \quad (19)$$

and

$$S_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^4x \sqrt{-g} R^2. \quad (20)$$

We calculate the tensor Q_{ab} , S_{ab} , and G_{ab} by recalling that for the metric (5) one has

$$R = 2(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + H_1^2 + H_2^2 + H_3^2 + H_1H_2 + H_2H_3 + H_1H_3) \quad (21)$$

and

$$R_{abcd}R^{abcd} = 4((\dot{H}_1 + H_1^2)^2 + (\dot{H}_2 + H_2^2)^2 + (\dot{H}_3 + H_3^2)^2 + H_1^2H_2^2 + H_1^2H_3^2 + H_2^2H_3^2). \quad (22)$$

We have

$$\frac{1}{a_1^2} G_{11} = \dot{H}_2 + \dot{H}_3 + H_2^2 + H_3^2 + H_2H_3 \quad (23)$$

$$\begin{aligned} \frac{1}{4a_1^2} Q_{11} = & 2\ddot{H}_1 + 4(H_1 + H_2 + H_3)\ddot{H}_1 + (3\dot{H}_1 + 2\dot{H}_2 + 2\dot{H}_3 - 2H_1^2 \\ & + 8H_1H_3 + 8H_1H_2 + 4H_2H_3)\dot{H}_1 + (\dot{H}_2 + 2H_1^2 + 2H_2^2 - 4H_1H_2)\dot{H}_2 \\ & + (\dot{H}_3 + 2H_1^2 + 2H_3^2 - 4H_1H_3)\dot{H}_3 - H_1^4 + H_2^4 + H_3^4 \\ & + H_1^2(H_2^2 + 4H_2H_3 + H_3^2) + H_2^2H_3^2 \\ & - 2H_1(H_2^3 + H_3^3 + H_2^2H_3 + H_2H_3^2) \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{4a_1^2} S_{11} = & 2(\ddot{H}_1 + \ddot{H}_2 + \ddot{H}_3) + 4(H_1 + H_2 + H_3)\ddot{H}_1 \\ & + 2(H_1 + 3H_2 + 2H_3)\ddot{H}_2 + 2(H_1 + 2H_2 + 3H_3)\ddot{H}_3 \\ & + (3\dot{H}_1 + 4(\dot{H}_2 + \dot{H}_3) - 2H_1^2 + 2(H_2 + H_3)^2 + 2H_1(H_2 + H_3))\dot{H}_1 \\ & + (5\dot{H}_2 + 6H_2^2 + 4H_3^2 + 2H_1(H_2 + H_3) + 8H_2H_3)\dot{H}_2 \\ & + (5\dot{H}_3 + 4H_2^2 + 6H_3^2 + 2H_1(H_2 + H_3) + 8H_2H_3)\dot{H}_3 + 6\dot{H}_2\dot{H}_3 \\ & - H_1^4 + H_2^4 + H_3^4 - 2(H_2 + H_3)H_1^3 + 2H_2^3H_3 + 2H_2H_3^3 \\ & + 3H_2^2H_3^2 - (H_2 + H_3)^2H_1^2. \end{aligned} \quad (25)$$

The other nonvanishing components $_{22}$ and $_{33}$ are obtained by index cyclic permutations from the above ones. One can check that in the isotropic case $H_1 = H_2 = H_3 = H$, we have $Q_{11} = Q_{22} = Q_{33} = 2\ddot{H} + 12H\dot{H} + 9(\dot{H} + 2H^2)\dot{H}$, and $S_{11} = S_{22} = S_{33} = 6\ddot{H} + 36H\dot{H} + 27(\dot{H} + 2H^2)\dot{H}$, in such a way that all the contributions from the higher order term cancel out in Einstein equations and we stay with the original equations of the model (Gunzig *et al.*, 2001a,b).

We notice that due to the term (22), we do not have anymore the first integral $q/r = (H_1 - H_2)/(H_2 - H_3) = \text{constant}$. This first integral was identified in (Abramo *et al.*, 2003a,b), and it is a consequence of the internal symmetry of Hilbert-Einstein action for Bianchi I metric (5) described in Chimento (2003). Contrary to the Kretschman scalar $R_{abcd}R^{abcd}$, the scalar curvature R is preserved under linear combinations of H_i that preserve the quantities $P = H_1 + H_2 + H_3$ and $S = H_1^2 + H_2^2 + H_3^2$. The intersection of constant P and S corresponds to a circumference in the Euclidean space of H_i , and its $SO(2)$ symmetry is responsible for the first integral (Saa, in preparation).

With the contribution from the tensors Q_{ab} and S_{ab} , the singular equations (12) and (13) are replaced by the regular ones

$$\alpha(4\ddot{q} + \dots) + F(\phi)\dot{q} = (F(\phi)p + F'(\phi)\dot{\phi})q, \tag{26}$$

$$\alpha(4\ddot{r} + \dots) + F(\phi)\dot{r} = (F(\phi)p + F'(\phi)\dot{\phi})r. \tag{27}$$

Both equations (26) and (27) are free from singularities on the surface $F(\phi) = 0$. With the hypothesis of small α , the new terms are relevant only when $F(\phi)$ vanishes. They assure a regular behavior of the solutions $q(t)$ and $r(t)$, eliminating the singular behavior present in the original equations (26) and (27).

5. CONCLUSION

We have shown that the inclusion of the conformal counterterm $C_{abcd}C^{abcd}$ in the action (1) can eliminate the anisotropic singularity corresponding to $F(\phi) = 0$. Such singularity is present in the recently proposed quintessential model (Gunzig *et al.*, 2001a,b), and it was its strongest objection. The conformal properties of the Weyl tensor C_{abcd} ensure that all isotropic solutions of (1) are preserved when the counterterm is included. Hence, all the interesting dynamical behavior described in Gunzig *et al.* (2001a,b) is still valid, including a cosmological history suitable, in principle, for describing quintessence.

A relevant issue is the naturalness of the necessary adjust in the constants α_1 , α_2 , and α_3 in order to have the conformal counterterm. As these constants may arise from quantum corrections, the only reasonable hypothesis about them is that

they must be small if compared with $1/G$. This point is now under investigation (Gunzig and Saa, in preparation), and preliminary results show that, under the only hypothesis of small α_1 , α_2 , and α_3 , both singularities on $F(\phi) = 0$ and $F_1(\phi) = 0$ can be eliminated preserving almost all of the isotropic behavior.

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